What is a Low Order Model?

$$\frac{\partial}{\partial t}\boldsymbol{\Psi} = \boldsymbol{NL}(\boldsymbol{\Psi}),$$

where ***NL*** is a nonlinear operator (quadratic nonlinearity)

$$\boldsymbol{\Psi}(x,y,z,...,t)=\sum_{i=-N}^{N} A_i(t)\Phi_i(x,y,z,...)$$

$$\frac{dA_i}{dt} = \sum_{j;k=-N}^{N} c_{ijk}A_jA_k \quad + \quad \sum_{j=-N}^{N} b_{ij}A_j + f_i \equiv v_i$$

Now truncate to some small N (≈ 3-30)!

Some useful properties:

(a) *Liouville Property*

If $c_{iij}=c_{iji}=0$, then $\Sigma(\partial v_i/\partial A_i) = 0$ and nonlinear terms preserve volume in phase space; rate of contraction of volume is $Tr(\mathbb{b})$.

(b) *Conservation Laws*

If special conditions on $\mathbb{c}$ are met, then nonlinear terms conserve $\Sigma\, g_i A_i^2$ for some vectors $\mathbb{g}$. (Typically one or two such vectors, corresponding to energy and enstrophy).

General Philosophy

"As I began to learn meteorology, I found it necessary to unlearn some mathematics." --Lorenz, Crafoord Prize Lecture 1983.

"There is virtually no limit to the number of phenomena which one might study by means of equations simplified according to the manner we have described. In each case, the simplified equations may seem to be rather crude approximations, but they should clarify our understanding of the phenomena, and lead to plausible hypotheses, *which may then be tested by means of careful observational studies and more refined systems of dynamic equations*." --Lorenz, *Tellus* 1960

"Very-low-order models cannot have as their purpose the quantitative duplication of real atmospheric behavior. Qualitatively they must reproduce some aspects of the behavior, if they are to serve any purpose. Often they are of pedagogical value; they can illustrate in an understandable manner the chain of events responsible for some phenomenon. Their chief use, however, may be exploratory; they can uncover new features or phenomena, *which can subsequently be checked with more detailed models, or perhaps with real observations*." --Lorenz, *JAS* 1984

Virtues of Low Order Models

- ♦ Easy to program, manage and modify
- ♦ Easy to archive and analyze output
- ♦ Rapidly generate data to test data analysis schemes
- ♦ May admit analytic treatment
- ♦ Possibility to analyze geometry of attractor
- ♦ Provide a quick check of physical reasoning
- ♦ Modest computer requirements

28-eqn. model, circa 1965:
1 cpu sec/time step (IBM 7090)

Contemporaneous models:
Phillips' 544 eqn 2-Layer QG model
GFDL 5184 eqn PE model

GFDL R15 9-level GCM, circa 1987 (32,400 eqn.):
.7 cpu sec/time step (CYBER 205)
(with full physics!)

Limiting factor is not computer power, but our ability to understand nonlinear systems.

Maximal Simplification:
The Basic Triad Equations

♦Truncate barotropic f-plane equations to 3 waves with $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0$ (Lorenz, *Tellus* 1960 and followers)

$$\frac{dA_1}{dt} = A_2 A_3 \quad - d_1 A_1$$
$$\frac{dA_2}{dt} = -2 A_3 A_1 + g\, A_2 + f\cos(\omega t)$$
$$\frac{dA_3}{dt} = A_2 A_1 \quad - d_3 A_3$$

These equations also describe *resonant* Rossby wave triads on the β-plane. ($\omega_1+\omega_2+\omega_3=0$)

Nonlinear terms conserve:

$$E = (A_1)^2+(A_2)^2+(A_3)^2$$
$$F = (A_1)^2 - (A_3)^2$$

Important ideas:

- ♦First concrete example of 2D cascade.
- ♦Nonlinear saturation of barotropic instability.
- ♦Associated mean flow oscillations.
- ♦Rossby waves are unstable (cf Lorenz*JAS* 1972 Hoskins*QJRMS* 1973, and Gill*GAFD* 1974.).
- ♦Variant: WMF interaction with mountains ⇒ orographic instability & multiple equilibria.

Wave-Mean-Flow Interactions in the Presence of Mountains

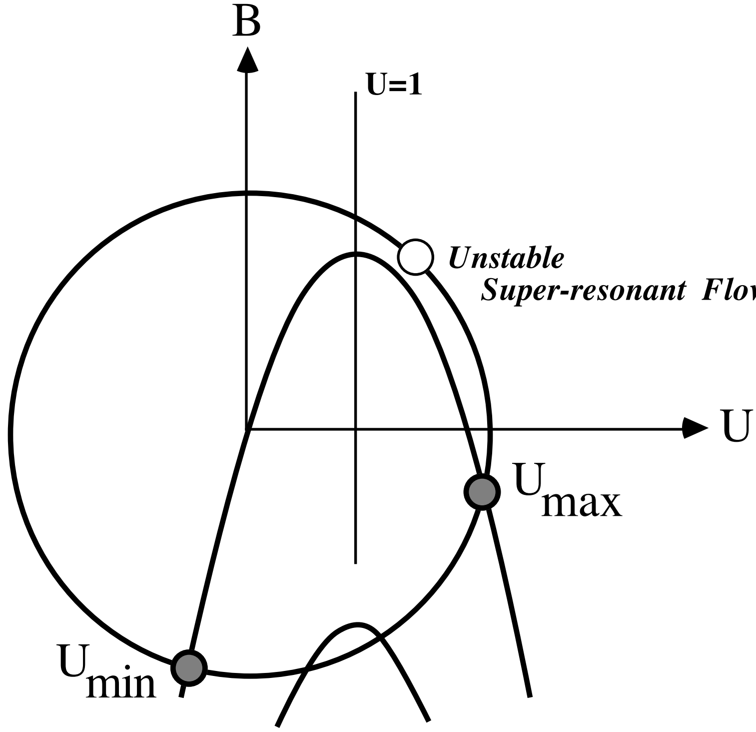
Barotropic β-plane truncated to 1 wave (Acos(x)+Bsin(x)) + mean flow U, with mountain (h cos(x)).

$$\frac{dA}{dt} = -(U-1)B + Uh - [A/\tau] \qquad \frac{dB}{dt} = (U-1)A - [B/\tau]$$

$$\frac{dU}{dt} = -\,Ah + [\kappa(U^{*}-U)]$$

Inviscid terms conserve:

$$E = A^2 + B^2 + U^2 \qquad F = Bh + \frac{1}{2}(U\ -1)^2$$



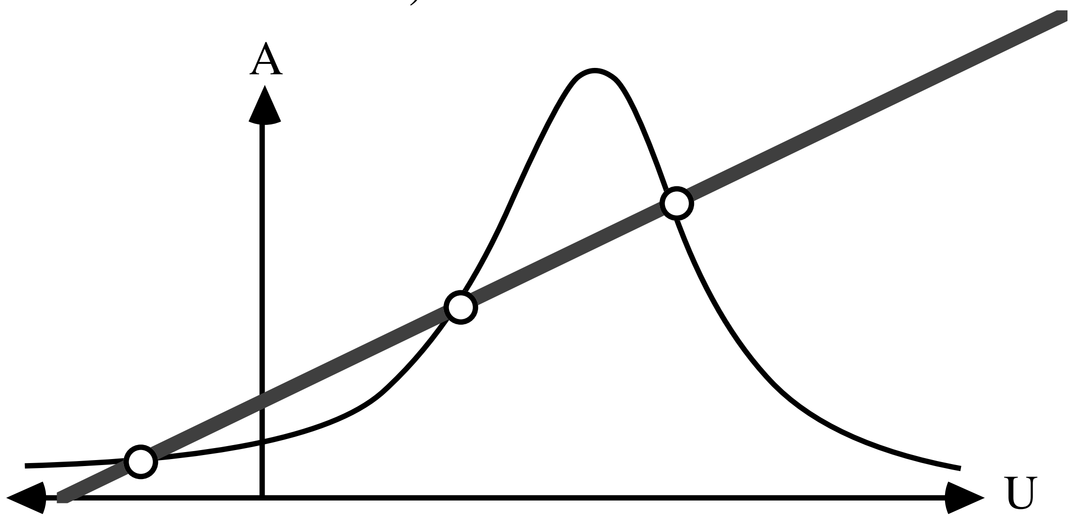
Important ideas:

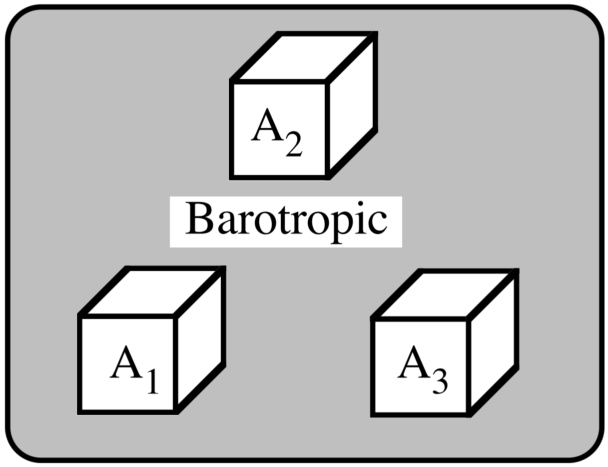
♦Orographic instability

♦Mean zonal wind reduction by mountain

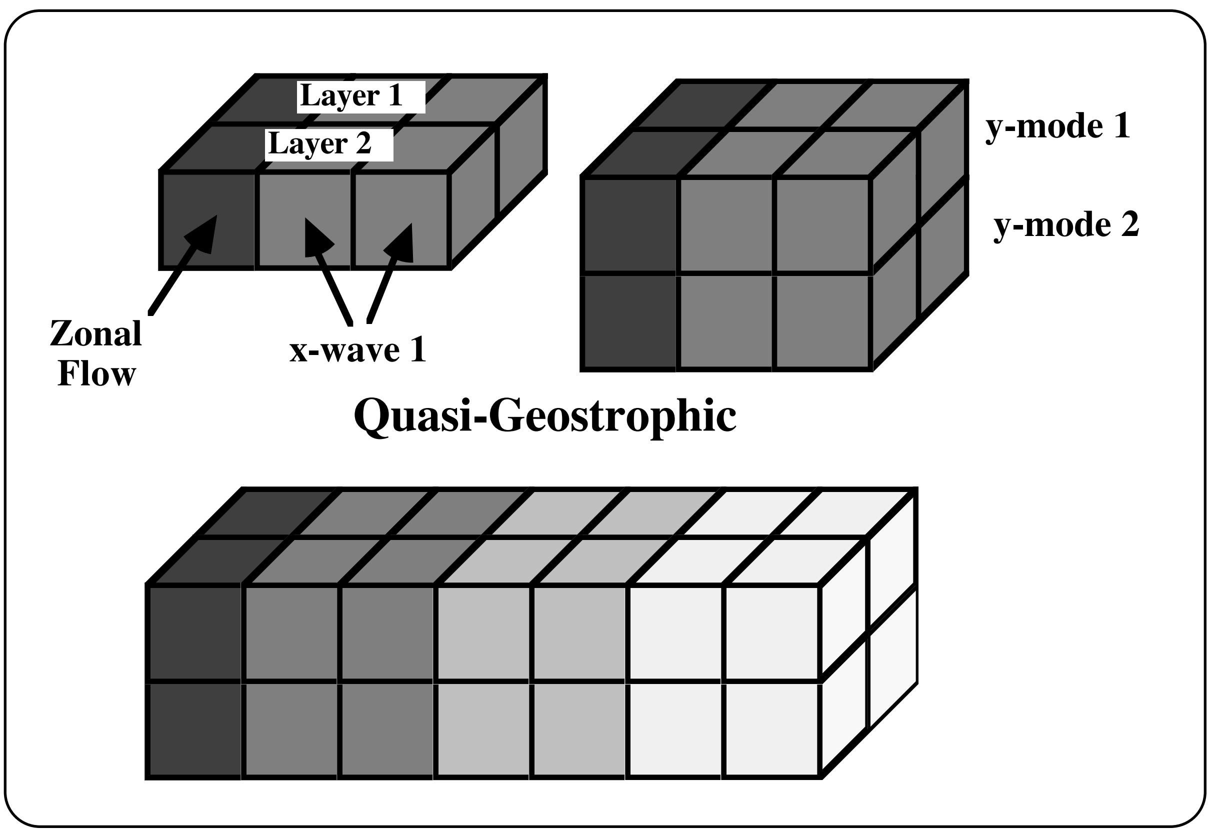
♦Zonal wind fluctuations associated with interference between standing and travelling wave.

With dissipative terms, get multiple equilibria (cf Charney & Devore and followers):

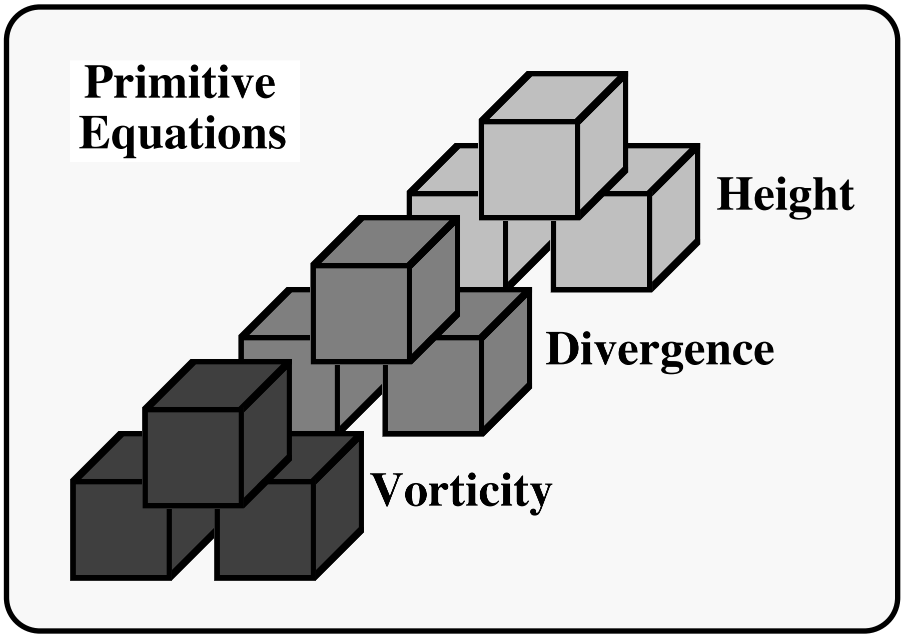




A_2
Barotropic
A_1
A_3

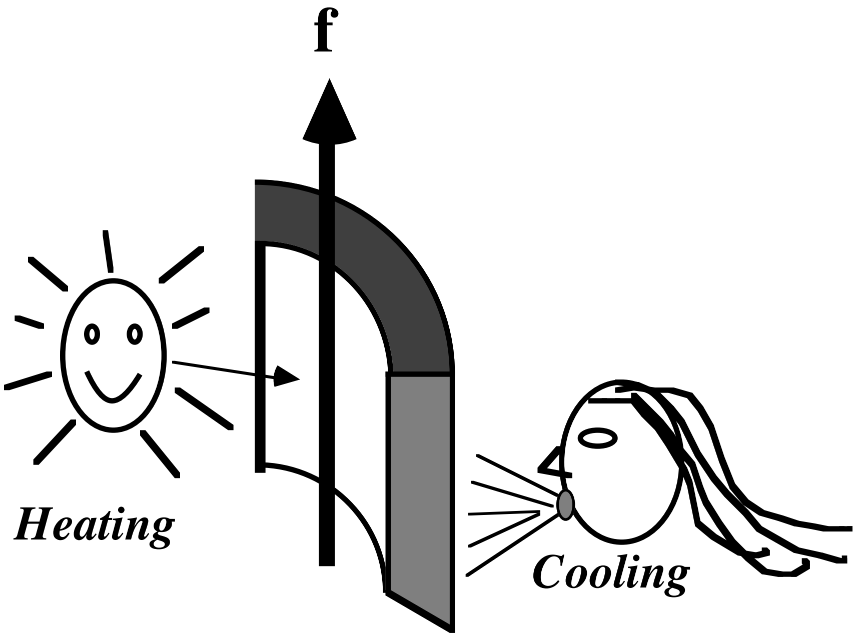


Layer 1
Layer 2
y-mode 1
y-mode 2
Zonal Flow
x-wave 1
Quasi-Geostrophic


Primitive
Equations
Height
Divergence
Vorticity

Mechanics of Vacillation

(Lorenz *JAS* 1962,1963)



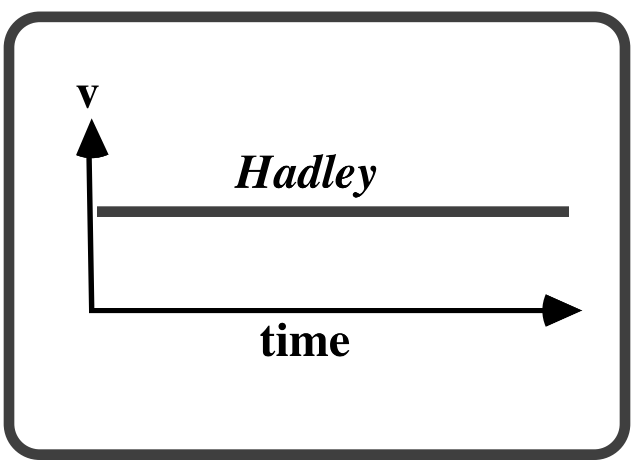
♦ 2-layer quasi-geostrophic channel model

♦ Mean flow + 1 wave in x; gravest mode in y ⇒ 6 eqn. model. (+ 2 for variable static stability).

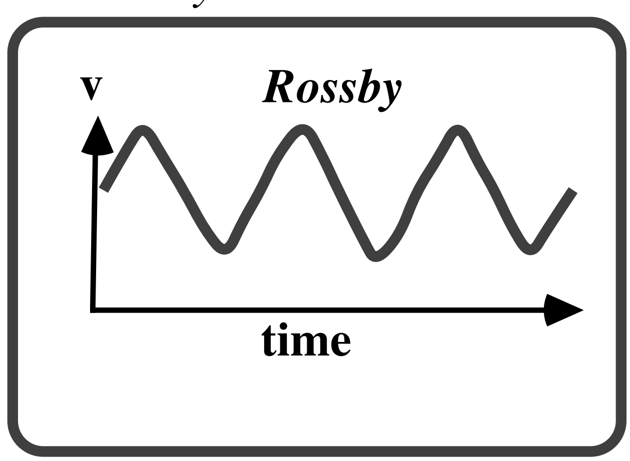
♦ Inclusion of second y-mode ⇒ 14 eqn. model.

8-eqn. model:

Fixed Point

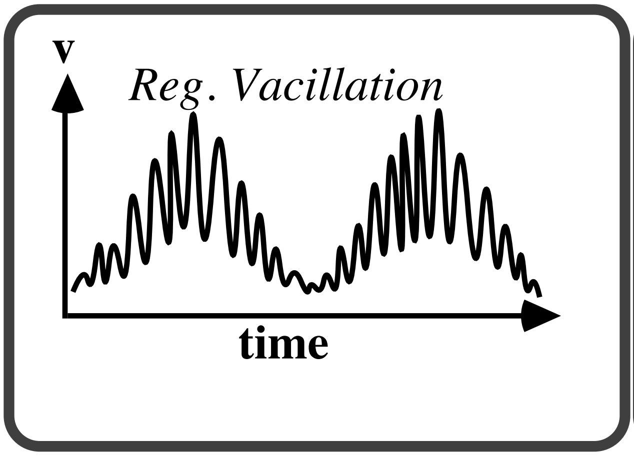


Limit Cycle

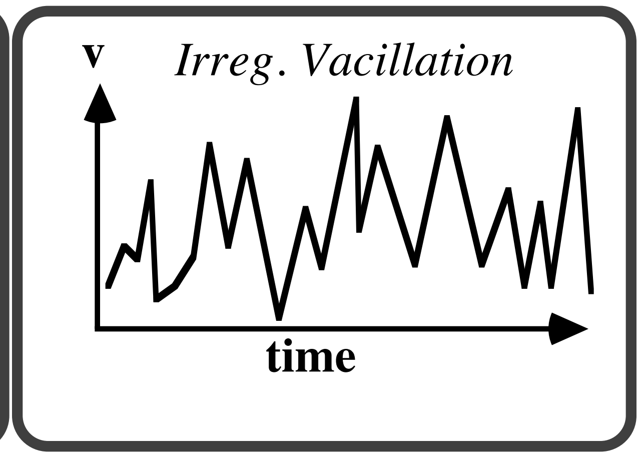


14-eqn. model:

2D Torus



Strange Attractor (?)



Important ideas:

♦ First example of nonlinear equilibration of baroclinic instability. Can equilibrate as travelling wave with steady amplitude. Limit cycle behavior.

♦ Regular vacillation can arise from a secondary instability of the travelling wave to a second degree of freedom in the cross-channel direction. Limit cycle behavior of wave *amplitude* .

♦ Irregular vacillation can result from instability of the aforementioned limit cycle.

Unresolved questions:

♦ Is irregular vacillation a high-order torus or a strange attractor?

♦ Effect of topography on vacillation.

Predictability

(Lorenz *Tellus* 1965,1969)

- ♦ 2-layer quasi-geostrophic channel model
- ♦ Mean flow+3 waves in x; 2 modes in y⇒28 eqns.

Method of analysis: *Predictability matrices*

(1) Let A (t) ≡ [A_1(t),...,A_N(t)] be the trajectory of the system.

(2) Linearize the system about the trajectory A (t). (can be done numerically with N perturbed integrations)

(3) Yields a *predictability matrix* with the property that $A(t) = \mathbb{M}(t,t_1)A(t_1)$

(4) The eigenvalues of $\mathbb{M}$ give the error growth; the fastest growing mode determines predictability time.

If the unperturbed trajectory is a steady state, method reduces to conventional stability analysis

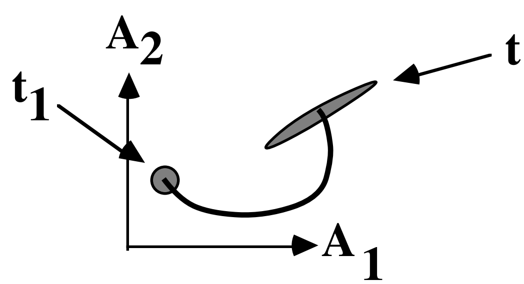


Illustration: *Lorenz 3-eqn "GCM"* *(Tellus* 1984*)*

$$\frac{dX}{dt} = -Y^2 - Z^2 - a X + aF$$

$$\frac{dY}{dt} = XY - b XZ - Y + G$$

$$\frac{dZ}{dt} = bXY + XZ - Z$$

Important ideas:

- ♦ First estimate of limit of deterministic prediction.
- ♦ Baroclinic instability ⇒ predictability loss
- ♦ "Identical twin" methodology
- ♦ Variability of predictability
- ♦ Persistent regimes
- ♦ Number of unstable directions << 28
- ♦ Importance of Lyapunov exponents.

Unresolved questions:

♦ What gross features of flow determine rate of error growth in the linear stage?

♦ Do time averages or other properties have extended predictability (i.e. past the linear growth stage)?

♦ Is predictability loss of the large-scale component due mainly to large-scale instability or to upscale nonlinear effects of synoptic scale predictability loss?

♦ What is the dimensionality of attractor for the 28-eqn. model, and how does it compare with the number of unstable directions?

The Slow Manifold and Initialization

(Lorenz *JAS* 1980,1986)

To initialize primitive equations, specify vorticity (Z), divergence (D) and geopotential (Φ) at each point$\Rightarrow$ 3M degrees of freedom. Fast gravity waves possible.

versus

To initialize quasigeostrophic equations, specify only geopotential (Φ) at each point$\Rightarrow$ M degrees of freedom. Gravity waves filtered out.

In QG flow, 2M of the 3M variables are slaved to the remaining M variables, defining a <M dimensional *slow manifold*. Does this happen in more general circumstances? How can we locate the slow manifold?

Basic idea: Instead of setting F_i=0, set dF_i/dt=0, where F_i is the "fast variable" amplitude. (cf Machenhauer, Baer-Tribbia)

The model:

♦ 1-layer primitive equation system

♦ Truncate to (1 wave in x)*(1 mode in y)*(3 fields) ⇒ 9 eqn. PE model with 3D quasigeostrophic subsystem.

Illustration: *5-eqn simplified PE model* (*JAS* 1986)

$$\frac{dU}{dt} = -VW + bVZ$$

$$\frac{dV}{dt} = UW - bUZ$$

$$\frac{dW}{dt} = -UV$$

$$\frac{dX}{dt} = -Z \qquad \frac{dZ}{dt} = X + b\,UV$$

Conserves $U^2 + V^2 + W^2 + X^2 + Z^2$ and $U^2 + V^2$.

⇒3D phase space $[\tan^{-1}(V/U), W, X]$

Important ideas:

♦ Slow manifold concept is valid

♦ Nonlinear normal mode initialization can indeed locate a slow manifold.

♦ Convergence of "superbalance" series not necessary for existence of a useful slow manifold.

Possible extensions:

♦ Use 9-eqn PE model to test data assimilation schemes (e.g. adjoint method).

♦ Explore diabatic initialization schemes (esp. effect of initialization on Hadley cell).

♦ Generation of gravity waves when balance equation becomes insoluble.

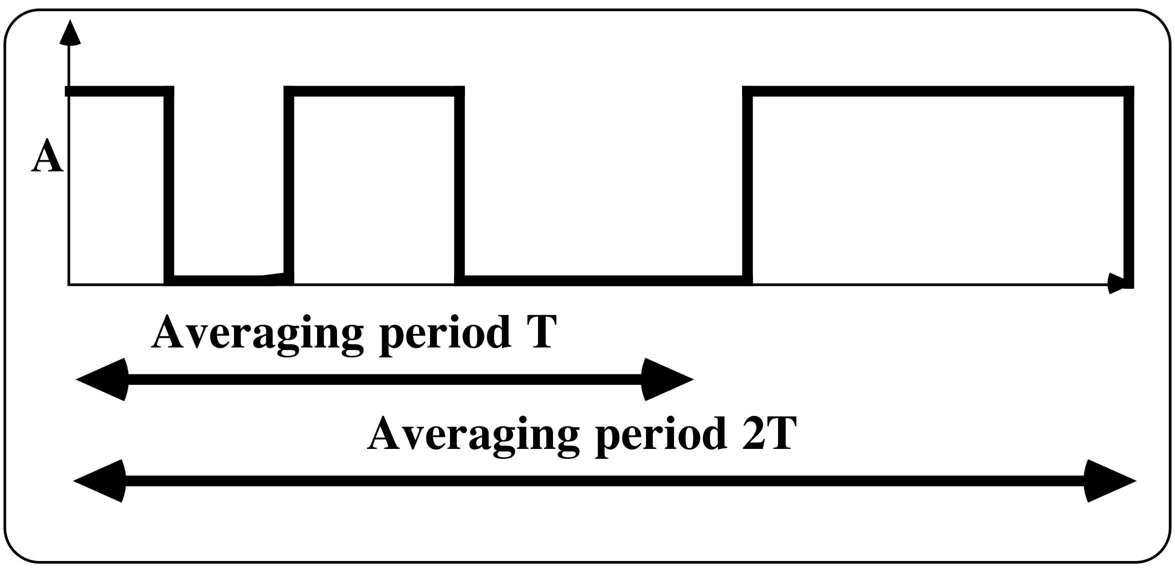
Low Frequency Variability

(a.k.a. weather regimes, "blocking," almost-intransitivity, etc.)

What is the nature of low frequency variability? How predictable is it?

Does Climate Exist?

The time average $\frac{1}{T}\int_0^T A(t)$ does not necessarily converge as $T \to \infty$. Failure is associated with high probability of persistent events (e.g. 1/f noise).



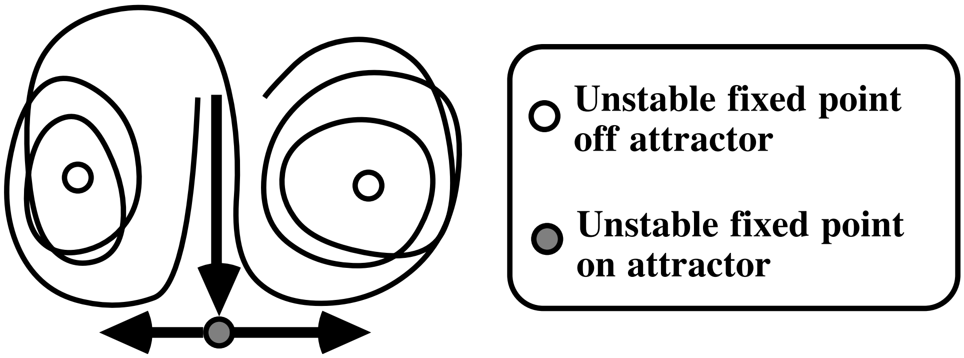
Is it unique?

i.e., with forcing and dissipation included, does long term behavior depend on initial conditions? (intransitivity) Or is the attractor all one piece?

Examples: Multiple stable fixed points or limit cycles (not very "atmospheric").

Persistent Structures:

(a) *Unstable* fixed points embedded in attractor, e.g. Hadley solution in 9-eqn. PE model. (Stable manifold must be non-empty).



(b) Unstable low-period orbits
(c) Unstable lower-dimension strange attractors
(d) Complex fixed points, etc. near the real axis.

How can we predict the presence of persistent structures?
How can we diagnose their presence in data?

Time-smoothed equations: an approach to regimes

Let B_i be "large scale" variables and A_i be "small scale" variables with $\langle A_i \rangle \approx 0$. Equations for $\frac{d\langle B_i \rangle}{dt}$ involve correlations $R_{ij} = \langle A_i A_j \rangle$, etc.

To what extent is R_{ij} a function of $\langle B_1 \rangle \ldots \langle B_M \rangle$?

Answer can be found by looking at geometry of time-smoothed trajectories.

Some other related issues:

- ♦ Probability distribution on attractor
- ♦ Persistence "" ""
- ♦ Recurrence times
- ♦ Relation of instability of climatological wave to structure of the large-scale variability. (cf Simmons, Branstator & Wallace).

Spatial Chaos

Equations for advection of a marker particle:

$$\frac{dX}{dt} = \frac{\partial\Psi(X,Y,t)}{\partial Y} \qquad \frac{dY}{dt} = -\frac{\partial\Psi(X,Y,t)}{\partial X}$$

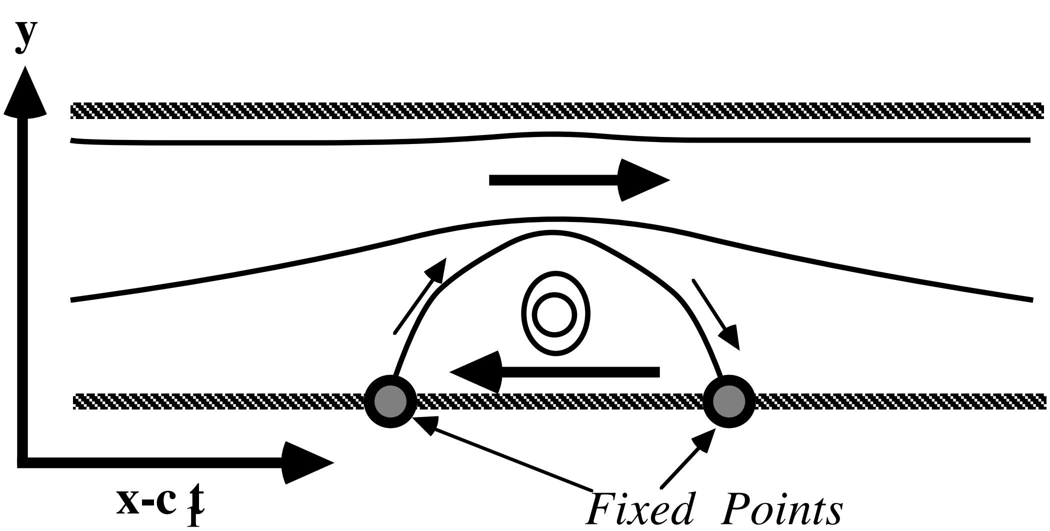
Simple flow fields can lead to chaotic advection!

Example: Two Rossby waves in a channel
(collaboration with A. Belmonte)

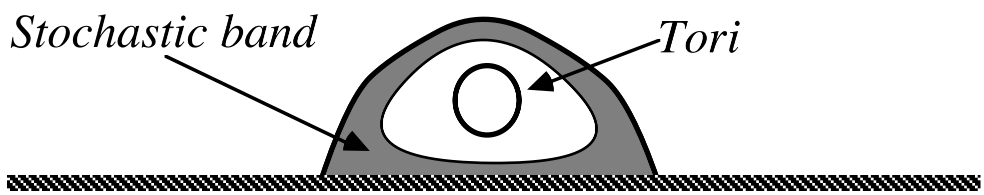
$$\Psi = [\ A\cos(k_1(x-c_1t)) + \varepsilon\cos(k_2(x-c_2t))\ ]\sin(y)$$

Use $[(x-c_1t), Y, k_2(c_1-c_2)t]$ as 3D phase space.

Streamlines of unperturbed flow:



Under weak perturbation:



Some questions:

- How much of mixing is due to large scale advection?
- What is the mixing pattern of common L.O.M.'s?
- How does this compare with high-resolution models?

N.B.-- Potential vorticity is also a tracer. Does potential vorticity homogenization in high-resolution models coincide with stochastic bands of L.O.M.'s?

The First 30 Years

Model	N	Phenomena
barotropic triad (+ forcing & dissip)	3	2D "cascade" Rossby wave instability Barotropic instability & mean flow oscillation Multiple equilibria Orographic instability
1 wave/1 mode 2 layer QG (+ mountains)	6	Nonlinear baroclinic instability
1 wave/2 mode 2 layer QG (+ mountains)	14	Regular/irregular vacillation Instability of tori
2-3 wave/2 mode 2 layer QG (+ mountains)	20 28	Predictability Scale interactions Low frequency variability
1 wave/2 mode moist 2 layer QG with mixed-layer ocean.	28	Midlatitude air-sea interaction SST effects on persistence Latent heating in storm tracks
1 wave/1 mode 1 layer PE	9	Slow manifold studies Initialization and assimilation Gravity wave generation
Simplified PE	5	"" "" ""
"Lorenz '63" model	3	Deterministic chaos, free will, etc. (but not convection!)
Littlest GCM	3	Predictability Almost-intransitivity Low freq. variability